Numerical Method - 2071

Q. 1. Convergence of Bisection method :-

In Bisection method, interval is halved every (iteration) interval. After n^{th} iteration, size of interval is reduced to:

$$\Delta n = \frac{(x_u - x_l)}{2^n}$$

Now we can say that maximum error after n^{th} iteration is

$$\varepsilon_n = \pm \Delta n$$

$$\Rightarrow |\varepsilon_n| = \frac{(x_u - x_l)}{2^n}$$

Similarly, after $(n+1)^{th}$ iteration, maximum error is given by,

$$\varepsilon_{n+1} = \frac{(x_u - x_l)}{2^{n+1}} = \frac{(x_u - x_l)}{2^n} \cdot \frac{1}{2} = \frac{\varepsilon_n}{2} \quad \text{--- (1)}$$

This eqn shows that error is halved after each iteration of bisection method, therefore we can say that bisection method converges linearly.

Soln:-

$$f(x) = x \tan x - 1 = 0$$

Let $x_1 = 7$, $x_2 = 8$ & $\varepsilon = 0.01$

Iteration 1:

$$f1 = f(x_1) = f(7) = -0.14$$

$$f2 = f(x_2) = f(8) = 0.124$$

Since, $f_1 \times f_2 < 0$, so it consists some roots.

$$x_0 = \frac{x_1 + x_2}{2} = \frac{7+8}{2} = 7.5$$

$$f_0 = f(x_0) = f(7.5) = -0.0126$$

Here, $f_1 \times f_0 > 0$, so

set $x_1 = x_0 = 7.5$

set $f_1 = f_0 = -0.0126$

$$\left|\frac{x_2 - x_1}{x_2}\right| = \left|\frac{8 - 7.5}{8}\right| = 0.0625 > \varepsilon$$

<u>Iteration 2:</u>

$$x_0 = \frac{x_1 + x_2}{2} = \frac{7.5 + 8}{2} = 7.75$$

$$f_0 = f(x_0) = f(7.75) = 0.0547$$

Here, $f_1 \times f_0 < 0$, so

set $x_2 = x_0 = 7.75$

set $f_2 = f_0 = 0.0547$

$$\left|\frac{x_2 - x_1}{x_2}\right| = \left|\frac{7.75 - 7.5}{7.75}\right| = 0.032 > \varepsilon$$

<u>Iteration 3:</u>

$$x_0 = \frac{x_1 + x_2}{2} = \frac{7.5 + 7.75}{2} = 7.625$$

$$f_0 = f(x_0) = f(7.625) = 0.02$$

Here, $f_0 \times f_1 < 0$, so

set $x_2 = x_0 = 7.625$

set $f_2 = f_0 = 0.02$

$$\left|\frac{x_2 - x_1}{x_2}\right| = \left|\frac{7.625 - 7.5}{7.625}\right| = 0.016 > \varepsilon$$

<u>Iteration 4:</u>

$$x_0 = \frac{x_1 + x_2}{2} = \frac{7.5 + 7.625}{2} = 7.5625$$

$$f_0 = f(x_0) = f(7.5625) = 0.004$$

Here, $f_1 \times f_0 < 0$, so

set $x_2 = x_0 = 7.5625$

set $f_2 = f_0 = 0.004$

$$\left|\frac{x_2 - x_1}{x_2}\right| = \left|\frac{7.5625 - 7.5}{7.5625}\right| = 0.008 < \varepsilon$$

$$\therefore \text{Root} = \frac{x_1 + x_2}{2} = \frac{7.5 + 7.5625}{2} = 7.53125$$

Q.2

x_i	0	1	2	3
e^{x_i}	0	1.7183	6.3891	19.0855

; $e^{1.9} = ?$

Soln:-

$x = 1.9$

$x_0 = 0 \qquad f_0 = 0$

$x_1 = 1 \qquad f_1 = 1.7183$

$x_2 = 2 \qquad f_2 = 6.3891$

Now,

$$l_0(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)$$

$$\text{or, } l_0(1.9) = \left(\frac{1.9 - 1}{0 - 1}\right)\left(\frac{1.9 - 2}{0 - 2}\right) = (-0.9) \times 0.05 = -0.045$$

$$l_1(x) = \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)$$

$$\text{or } l_1(1.9) = \left(\frac{1.9 - 0}{1 - 0}\right)\left(\frac{1.9 - 2}{1 - 2}\right) = 1.9 \times 0.1 = 0.19$$

$$l_2(x) = \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)$$

$$\text{or } l_2(1.9) = \left(\frac{1.9 - 0}{2 - 0}\right)\left(\frac{1.9 - 1}{2 - 1}\right) = 0.95 \times 0.9 = 0.855$$

Now we have, from 2nd order Lagrange Interpolation.

$$P_2(1.9) = f_0 l_0(1.9) + f_1 l_1(1.9) + f_2 l_2(1.9)$$

$$= 0 \times (-0.045) + 1.7183 \times 0.19 + 6.3891 \times 0.855$$

$$= 5.789$$

Q.3 $\quad I = \int_0^2 \left(e^{x^2} - 1\right) dx$ with $n = 8 = k$

Here, $x_0 = 0$, $x_n = 2$, $h = \frac{x_n - x_0}{n} = \frac{2 - 0}{8} = 0.25$

$f(x) = e^{x^2} - 1$

$f(x_0) = f(0) = 0$

$f(x_n) = f(2) = 53.59$

$f(x_1) = f(x_0+h) = f(0.25) = 0.064$

$f(x_2) = f(x_0+2h) = f(0.5) = 0.284$

$f(x_3) = f(x_0+3h) = f(0.75) = 0.755$

$f(x_4) = f(x_0+4h) = f(1) = 1.718$

$f(x_5) = f(x_0+5h) = f(1.25) = 3.77$

$f(x_6) = f(x_0+6h) = f(1.5) = 8.487$

$f(x_7) = f(x_0+7h) = f(1.75) = 20.38$

Now, from Simpson's composite 1/3 Rule, we have,

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4\{f(x_1)+f(x_3)+f(x_5)+f(x_7)\} + 2\{f(x_2)+f(x_4)+f(x_6)\} + f(x_n)\right]$$

$$\therefore \int_0^2 (e^{x^2}-1)\,dx = \frac{0.25}{3}\left[0 + 4\{0.064+0.755+3.77+20.38\} + 2\{0.284+1.718+8.487\} + 53.59\right]$$

$$= 0.0833\,[4\times 24.969 + 2\times 10.489 + 53.59]$$

$$= 14.53$$

Q 4. Certain systems of linear equations are such that their solutions are very sensitive to small changes (and therefore to errors) in their coefficients & constants. Eg: If 1% changes in two coefficients change the solution by a factor of 10 or more. Such systems are said to be ill-conditioned.

Solⁿ –

Writing the eqⁿs in the form:

$$Ax = C$$

or
$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

So,
$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

we have,

$$A = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$u_{11} = a_{11} = 3$

$u_{12} = a_{12} = 2$

$u_{13} = a_{13} = 1$

$l_{21} = \dfrac{a_{21}}{a_{11}} = \dfrac{2}{3} = 0.67$

$u_{22} = a_{22} - l_{21}u_{12} = 3 - 0.67 \times 2 = 1.67$

$u_{23} = a_{23} - l_{21}u_{13} = 2 - 0.67 \times 1 = 1.33$

$l_{31} = \dfrac{a_{31}}{u_{11}} = \dfrac{1}{3} = 0.33$

$l_{32} = \dfrac{a_{32} - l_{31}u_{12}}{u_{22}} = \dfrac{2 - 0.33 \times 2}{1.67} = 0.8$

$u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$

$= 3 - 0.33 \times 1 - 0.8 \times 1.33$

$= 1.606$

$$\therefore [A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 0.67 & 1 & 0 \\ 0.33 & 0.8 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1.67 & 1.33 \\ 0 & 0 & 1.606 \end{bmatrix}$$

Now, we have,

$LZ = C$ where, $Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

or, $\begin{bmatrix} 1 & 0 & 0 \\ 0.67 & 1 & 0 \\ 0.33 & 0.8 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$

The corresponding system is:

$z_1 = 10 \quad —(1)$

$0.67z_1 + z_2 = 14 \quad —(2)$

$0.33z_1 + 0.8z_2 + z_3 = 14 \quad —(3)$

from ① & ②

$0.67 \times 10 + z_2 = 14$

or $z_2 = 7.3$

from ③

$0.33 \times 10 + 0.8 \times 7.3 + z_3 = 14$

or $z_3 = 4.86$

$\therefore \; Z = \begin{bmatrix} 10 \\ 7.3 \\ 4.86 \end{bmatrix}$

Again, $UX = Z$

or $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1.67 & 1.33 \\ 0 & 0 & 1.606 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 7.3 \\ 4.86 \end{bmatrix}$

The corresponding system is :

$3x_1 + 2x_2 + x_3 = 10$ — ④

$1.67x_2 + 1.33x_3 = 7.3$ — ⑤

$1.606x_3 = 4.86$ — ⑥

from ⑥,

$x_3 = 3.02$

from ⑤ & ⑥,

$1.67x_2 + 1.33 \times 3.02 = 7.3$

or, $x_2 = 1.96$

from ④,

$3x_1 + 2 \times 1.96 + 3.02 = 10$

or $x_1 = 1.02$

$\therefore \; x_1 = 1.02 \approx 1$

$x_2 = 1.96 \approx 2$

$x_3 = 3.02 \approx 3$

Q.6. The finite difference formula for solving Poisson's Equation is:

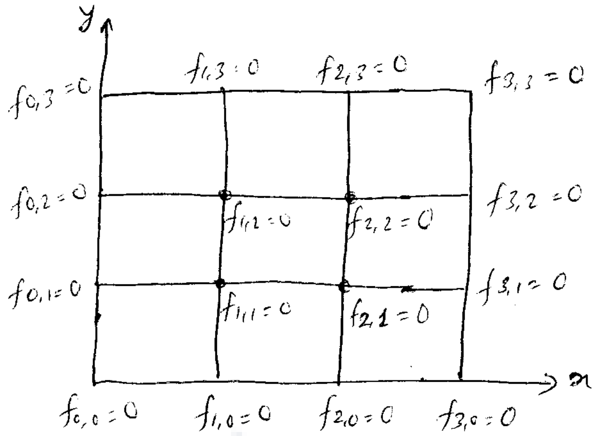
$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = h^2 g_{i,j}$$

<u>Soln:-</u>

Given Poissons eqn is $\nabla^2 f = 2x^2y^2$; $0 \le x \le 3$, $0 \le y \le 3$.

$h = 1$.

Lets divide the domain into grids of 3x3 as below:



At $i=1$ & $j=1$,

$$f_{2,1} + f_{0,1} + f_{1,2} + f_{1,0} - 4f_{1,1} = 2$$

or, $f_{2,1} + f_{1,2} - 4f_{1,1} = 2$ --- (1)

At $i=2$ and $j=1$,

$$f_{3,1} + f_{1,1} + f_{2,2} + f_{2,0} - 4f_{2,1} = 8$$

or, $f_{1,1} + f_{2,2} - 4f_{2,1} = 8$ --- (2)

At $i=1$ & $j=2$,

$$f_{2,2} + f_{0,2} + f_{1,3} + f_{1,1} - 4f_{1,2} = 8$$

or, $f_{2,2} + f_{1,1} - 4f_{1,2} = 8$ --- (3)

At $i=2$ & $j=2$,

$$f_{3,2} + f_{1,2} + f_{2,3} + f_{2,1} - 4f_{2,2} = 32$$

or, $f_{1,2} + f_{2,1} - 4f_{2,2} = 32$ --- (4)

Solving eqns (1) to (4),

Using eqn (4) in (1)

$$f_{2,1} + 32 + 4f_{2,2} - \cancel{f_{2,1}} - 4f_{1,1} = 2$$

or $4f_{2,2} - 4f_{1,1} = -30$ --- (9)

Using eq^n (4) in (2),

$$f_{2,2} + f_{1,1} - 4[32 + 4f_{2,2} - f_{1,2}] = 8$$

or, $f_{2,2} + f_{1,1} - 128 - 16f_{2,2} + 4f_{1,2} = 8$

or, $-15f_{2,2} + f_{1,1} + 4f_{1,2} = 136$ ---- (b).

& we have, eq^n (3),

$$f_{2,2} + f_{1,1} - 4f_{1,2} = 8 \quad --- (3).$$

Solving eq^{n}s (a), (b) & (3), we get,

$$f_{2,2} = \frac{-43}{4}, \quad f_{1,1} = \frac{-13}{4}, \quad f_{1,2} = \frac{-11}{2}$$

Using these values in eq^n (1),

$$f_{2,1} - \frac{11}{2} - 4\left(\frac{-13}{4}\right) = 2$$

or $f_{2,1} = \frac{-11}{2}$

Q.7 <u>Algorithm for fixed Point method</u> :-

1) Convert the function $f(x) = 0$ in the form $x = g(x)$
2) Decide the initial values for x_0 & $\mathcal{E}$
3) Calculate $x_1 = g(x_0)$
4) If $\left|\frac{x_1 - x_0}{x_1}\right| < \mathcal{E}$, then root $= x_1$ and stop

   else

   set $x_0 = x_1$ and go to step 3.

<u>C Program for fixed Point method</u> :-

```
#include<stdio.h>
#include <conio.h>
#include <math.h>
#define G(x) (a3*x*x*x + a2*x*x + a0)/(-a1)
float a0, a1, a2, a3;
int main()
{
```

```
float x0, x1, E, Er;
printf ("Enter coefficients a3, a2, a1 & a0 \n");
scanf ("%f %f %f %f", &a3, &a2, &a1, &a0);
printf ("Enter initial guess & E\n");
scanf ("%f %f", &x0, &E);
while(1)
{
    x1 = G(x0);
    Er = (x1 - x0)/x1;
    if (fabs(Er)<E)
    {
        printf ("Root= %f \n", x1);
        break;
    }
    x0 = x1;
}
getch();
}
```

Output:-

```
Enter coefficients a3, a2, a1 & a0
1    0    -7    2
Enter initial guess and E
1    0.05

Root = 0.28945
```